

The non-planar contribution to the four-loop anomalous dimension of twist-2 operators: first moments in $\mathcal{N} = 4$ SYM and non-singlet QCD

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Abstract

We present the result of a *full direct* component calculation for the first three even moments of the non-planar contribution into the four-loop anomalous dimension of twist-2 operators in maximally extended $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Obtained result completes our previous calculations in arXiv:0902.4646 and gives the usual result for the higher moments on the contrary to degenerate one in the case of Konishi. We propose a general form of ζ_5 and ζ_3 parts of the complete non-planar four-loop anomalous dimension of twist-2 operators. As by product, we have obtained the first even moment of the non-planar contribution to the non-singlet four-loop anomalous dimension of Wilson twist-2 operators in QCD.

Calculation of anomalous dimensions of the Wilson twist-2 operators is one of the part of operator product expansion for the structure functions in the framework of perturbative Quantum Chromodynamics (QCD). At the present time such calculations are performed up to three-loop order [1], while other part of operator product expansion, the coefficient functions, which are known in the same order [2], demand the four-loop anomalous dimensions.

Moreover, the great interest in the calculations of anomalous dimensions of the composite operators comes from the investigations of integrability in the framework of AdS/CFT-correspondence [3]. In the planar limit there are different calculations [4, 5] for the test of Asymptotic Bethe Ansatz [6] as well as for recently proposed spectral equations for the finite length operators [7]. The calculations are performed perturbatively up to four/five loops [4] for twist-2/twist-3 operators and up to one loop more with the using of generalized Lüscher corrections [8] for the operators with arbitrary Lorentz spin [5].

For non-planar case we calculated some time ago the anomalous dimension of the Konishi operator at fourth order in $\mathcal{N} = 4$ SYM theory [9], where non-planar contribution appears for the first time for the twist-2 operators. The result was rather surprising, since it contains only ζ_5 contribution without ζ_3 and rational parts:

$$\gamma_{\text{Konishi}}^{4\text{-loop, np}} = -\frac{17280}{N_c^2} \zeta_5 g^8, \quad g^2 = \frac{g_{YM}^2 N_c}{(4\pi)^2}. \quad (1)$$

The calculations of the ζ_5 contribution to the first three even moments allowed us to assume the following general form of non-planar contribution to the anomalous dimension of twist-2 operators with arbitrary Lorentz spin:

$$\gamma_{\text{uni, np}}(j) = -640 S_1^2(j-2) \frac{12}{N_c^2} \zeta_5 g^8 + \dots, \quad S_1(j) = \sum_{i=1}^j \frac{1}{i}. \quad (2)$$

However, this result is in contradiction with the usual large- j behavior of anomalous dimension, which is expected to be proportional $\ln j$ (see Ref.[10]), while from Eq.(2) we obtain $(\ln j)^2$.

In this paper we present the result of calculations of non-planar contribution for the first three even moments of the four-loop anomalous dimension of twist-2 operators in $\mathcal{N} = 4$ SYM theory, which completes our result Eq.(1). Moreover, during these calculations we have obtained the first even moment of the non-planar contribution to the non-singlet anomalous dimension of Wilson twist-2 operator at fourth order in perturbative QCD. The result for the first even moment of the four-loop non-singlet anomalous dimension of Wilson twist-2 operator in QCD can be found in Ref.[11], but our result for the non-planar contribution contains full color and flavor structures and the calculations are performed with a different method¹.

¹Note, that there is all-loop prediction for the $\mathcal{O}(1/N_f)$ contribution to the non-singlet anomalous dimension of twist-2 operators in QCD [12].

The calculations were performed in the same way, as in our previous work [13]. We consider the following “QCD-like” colour and $SU(4)$ singlet local Wilson twist-2 operators:

$$\mathcal{O}_{\mu_1, \dots, \mu_j}^g = \hat{S} G_{\rho\mu_1}^a \mathcal{D}_{\mu_2} \mathcal{D}_{\mu_3} \dots \mathcal{D}_{\mu_{j-1}} G_{\rho\mu_j}^a, \quad (3)$$

$$\mathcal{O}_{\mu_1, \dots, \mu_j}^\lambda = \hat{S} \bar{\lambda}_i^a \gamma_{\mu_1} \mathcal{D}_{\mu_2} \dots \mathcal{D}_{\mu_j} \lambda^{a\ i}, \quad (4)$$

$$\mathcal{O}_{\mu_1, \dots, \mu_j}^\phi = \hat{S} \bar{\phi}_r^a \mathcal{D}_{\mu_1} \mathcal{D}_{\mu_2} \dots \mathcal{D}_{\mu_j} \phi_r^a, \quad (5)$$

where \mathcal{D}_{μ_i} are covariant derivatives. The spinors λ_i and field tensor $G_{\rho\mu}$ describe gauginos and gauge fields, respectively, and ϕ_r are the complex scalar fields appearing in the $\mathcal{N} = 4$ SYM theory. Indices $i = 1, \dots, 4$ and $r = 1, \dots, 3$ refer to $SU(4)$ and $SO(6) \simeq SU(4)$ groups of inner symmetry, respectively. The symbol \hat{S} implies a symmetrization of each tensor in the Lorentz indices μ_1, \dots, μ_j and a subtraction of its traces. These operators mix with each other under renormalization and the eigenvalues of the matrix are expressed through the universal anomalous dimension (see our papers in Ref.[4] for details)

$$\gamma_{uni}(j) = \sum_{n=0}^{\infty} \gamma_{uni}^{(n)}(j) g^{2(n+1)}. \quad (6)$$

We will be interested in the following leading order multiplicative renormalizable combinations of the operators (3)-(5) with $j=2$ (see details in Ref.[13])²:

$$\mathcal{O}_{\mu\nu}^T = \mathcal{O}_{\mu\nu}^g + \mathcal{O}_{\mu\nu}^\lambda + \mathcal{O}_{\mu\nu}^\phi, \quad (7)$$

$$\mathcal{O}_{\mu\nu}^\Sigma = -2 \mathcal{O}_{\mu\nu}^g + \mathcal{O}_{\mu\nu}^\lambda + 2 \mathcal{O}_{\mu\nu}^\phi, \quad (8)$$

$$\mathcal{O}_{\mu\nu}^\Xi = -\frac{1}{4} \mathcal{O}_{\mu\nu}^g + \mathcal{O}_{\mu\nu}^\lambda - \frac{3}{2} \mathcal{O}_{\mu\nu}^\phi. \quad (9)$$

Operator $\mathcal{O}_{\mu\nu}^T$ is the stress tensor. Its anomalous dimension is equal to zero and corresponds to $\gamma_{uni}^{(0)}(j=2)$. Operator $\mathcal{O}_{\mu\nu}^\Sigma$ has the same anomalous dimension as the Konishi operator, which corresponds to $\gamma_{uni}^{(0)}(j=4)$ and we know its anomalous dimension (see Eq.(1)). Operator $\mathcal{O}_{\mu\nu}^\Xi$ has the anomalous dimension, which corresponds to the value of universal anomalous dimension $\gamma_{uni}^{(0)}(j)$ with $j=6$.

In the next-to-leading order operators (7)-(9) mix with each other under renormalization. It is related with the breaking of the conformal invariance if we consider more general conformal operators [14]. Breaking of conformal invariance is controlled by the conformal Ward identity [15] (see also Ref.[16]), which allows obtain the results for the anomalous dimensions of the conformal operators in ℓ^{th} -loops order with additional $(\ell-1)$ -loops calculations [17].

If we go to the four loops we can do the same as in the leading order for the contribution to the universal anomalous dimension, which appears for the first time at this order because

²The coefficients in the front of the operators $\mathcal{O}_{\mu\nu}^g$, $\mathcal{O}_{\mu\nu}^\lambda$ and $\mathcal{O}_{\mu\nu}^\phi$ in Eqs.(7)-(9) are the same (up to common factor) as in the conformal operators $\Xi_{\mu\nu}$, $\Sigma_{\mu\nu}$ and $T_{\mu\nu}$ from the first paper in Ref.[4].

there are no additional contributions either from the renormalizations or from the conformal Ward identities. We used this property for the calculation of ζ_5 contribution and will use here for the calculation of the non-planar contribution.

Thus, we need to calculate the matrix elements for the operators $\mathcal{O}_{\mu\nu}^T$, $\mathcal{O}_{\mu\nu}^\Sigma$ and $\mathcal{O}_{\mu\nu}^\Xi$ sandwiched between fermion states (as the most simple case) and look only at the pole with quartic Casimir operator $d_{44} = N^2(N^2 + 36)/24$. This can be done with our program BAMBA based on the algorithm of Laporta [18] (see also Ref.[19, 20]), which we used in our previous calculations.

All calculations were performed with FORM [21], using FORM package COLOR [22] for evaluation of the color traces and with the Feynman rules from Refs.[23]. For the dealing with a huge number of diagrams we use a program DIANA [24], which call QGRAF [25] to generate all diagrams.

In fact, we have computed the non-planar contributions to the four-loop anomalous dimensions $\gamma_{g\lambda}$, $\gamma_{\phi\lambda}$ and $\gamma_{\lambda\lambda}$ of the operators $\mathcal{O}_{\mu\nu}^g$, $\mathcal{O}_{\mu\nu}^\lambda$ and $\mathcal{O}_{\mu\nu}^\phi$ sandwiched between the fermion states to have a possibility to combine them with the coefficients from Eqs.(7)-(9) for the additional check. We have obtained the following results for the non-planar contributions to the four-loop anomalous dimensions of the operators $\mathcal{O}_{\mu\nu}^T$, $\mathcal{O}_{\mu\nu}^\Sigma$ and $\mathcal{O}_{\mu\nu}^\Xi$ sandwiched between fermion states:

$$\Gamma_{T_{\mu\nu}}^{np} = 0, \quad (10)$$

$$\Gamma_{\Sigma_{\mu\nu}}^{np} = -360 \zeta_5 \frac{48 g^8}{N_c^2} + \mathcal{O}(g^{10}), \quad (11)$$

$$\Gamma_{\Xi_{\mu\nu}}^{np} = \frac{25}{9} \left(21 + 70 \zeta_3 - 250 \zeta_5 \right) \frac{48 g^8}{N_c^2} + \mathcal{O}(g^{10}). \quad (12)$$

The first two results coincide with the known results [9]. The ζ_5 part of the third result we already know [9] and we suggest the following general expression for the ζ_5 contribution to the non-planar four-loop universal anomalous dimension (see Ref.[9]):

$$\gamma_{uni,np,\zeta_5}^{(3)}(j) = -160 S_1^2(j-2) \quad (13)$$

with

$$\gamma_{uni,np}^{(3)}(j) = \left(\gamma_{uni,np,\zeta_5}^{(3)}(j) \zeta_5 + \gamma_{uni,np,\zeta_3}^{(3)}(j) \zeta_3 + \gamma_{uni,np,rational}^{(3)}(j) \right) \frac{48}{N_c^2}. \quad (14)$$

Now we can try to reconstruct a general form of the ζ_3 part of the non-planar contribution to the four-loop universal anomalous dimension $\gamma_{uni,np}^{(3)}$. Following to the principle of maximal transcendentality [26], in the fourth order of the perturbative theory the transcendentality level of the universal anomalous dimension is equal to 7 and the transcendentality of ζ_3 is equal to 3. Let's suppose, that the reciprocity [27] will hold also for the non-planar contribution. Thus, the basis for ansatz will consist of the binomial harmonic sums, defined

through (see Ref.[28] and our papers in Ref.[5] for details)

$$\mathbb{S}_{i_1, \dots, i_k}(N) = (-1)^N \sum_{j=1}^N (-1)^j \binom{N}{j} \binom{N+j}{j} S_{i_1, \dots, i_k}(j), \quad (15)$$

which should have transcendentality 4. In the above equation S_{i_1, \dots, i_k} are the harmonic sums [28]

$$S_a(N) = \sum_{j=1}^N \frac{(\text{sgn}(a))^j}{j^{|a|}}, \quad S_{a_1, \dots, a_n}(N) = \sum_{j=1}^N \frac{(\text{sgn}(a_1))^j}{j^{|a_1|}} S_{a_2, \dots, a_n}(j) \quad (16)$$

and the indices i_1, \dots, i_k are *positive*. There are $2^3 = 8$ such binomial harmonic sums:

$$\mathbb{S}_4, \mathbb{S}_{3,1}, \mathbb{S}_{2,2}, \mathbb{S}_{2,1,1}, \mathbb{S}_{1,3}, \mathbb{S}_{1,2,1}, \mathbb{S}_{1,1,2}, \mathbb{S}_{1,1,1,1}. \quad (17)$$

The basis from above binomial sums can be rewritten in the following equivalent form:

$$\mathbb{S}_4, \mathbb{S}_{3,1}, \mathbb{S}_{2,2}, \mathbb{S}_{2,1,1}, \mathbb{S}_1 \mathbb{S}_3, \mathbb{S}_1 \mathbb{S}_{2,1}, \mathbb{S}_1^2 \mathbb{S}_2, \mathbb{S}_1^4. \quad (18)$$

The common factor in $\Gamma_{\Xi_{\mu\nu}}$ Eq.(12) can be written as $4 S_1(4) = 2 \mathbb{S}_1(4)$. Therefore, let's try ansatz, which consists of the binomial harmonic sums in Eq.(18) with \mathbb{S}_1 except for the last sum:

$$\mathbb{S}_1 \mathbb{S}_3, \mathbb{S}_1 \mathbb{S}_{2,1}, \mathbb{S}_1^2 \mathbb{S}_2. \quad (19)$$

Note, that this basis contains the same binomial sums as the ζ_3 part of the planar contribution to the four-loop universal anomalous dimension

$$\gamma_{uni,pl,\zeta_3}^{(3)}(j) = 64 \mathbb{S}_1 (\mathbb{S}_3 - \mathbb{S}_{2,1}) + 64 \mathbb{S}_1^2 \mathbb{S}_2, \quad (20)$$

where the first term comes from the dressing phase while the second term comes from the wrapping corrections. Here and in the following the argument of the (binomial) harmonic sums is $j-2$. Really, we have only two non-trivial values (11) and (12), then the solution has one free integer parameter x

$$\gamma_{uni,np,\zeta_3}^{(3)}(j) = \mathbb{S}_1 (9(24-x) \mathbb{S}_3 + (x-72) \mathbb{S}_{2,1} + x \mathbb{S}_1 \mathbb{S}_2). \quad (21)$$

In principal, we can fix x with some reasonable conditions, for example:

1) putting the coefficient of $\mathbb{S}_1 \mathbb{S}_3$ equal to zero

$$\gamma_{uni,np,\zeta_3}^{(3)}(j) = 24 \mathbb{S}_1 (\mathbb{S}_1 \mathbb{S}_2 - 2 \mathbb{S}_{2,1}); \quad (22)$$

2) putting the coefficient of $\mathbb{S}_1 \mathbb{S}_{2,1}$ equal to zero

$$\gamma_{uni,np,\zeta_3}^{(3)}(j) = 72 \mathbb{S}_1 (\mathbb{S}_1 \mathbb{S}_2 - 6 \mathbb{S}_3); \quad (23)$$

3) putting the coefficient of $\mathbb{S}_1^2 \mathbb{S}_2$ equal to zero

$$\gamma_{uni,np,\zeta_3}^{(3)}(j) = 72 \mathbb{S}_1 (3 \mathbb{S}_3 - \mathbb{S}_{2,1}) ; \quad (24)$$

4) split final expression into the part, which is similar to wrapping corrections in the planar limit, and the remnant

$$\gamma_{uni,np,\zeta_3}^{(3)}(j) = 8 \mathbb{S}_1 (2 \mathbb{S}_1 \mathbb{S}_2 + 9 \mathbb{S}_3 - 7 \mathbb{S}_{2,1}) . \quad (25)$$

It is interesting that Eq.(22) gives zero at $j = 3$ as well as for $j = 2$ and $j = 4$. So, we believe, that Eq.(22) is the preferable expression for the ζ_3 contribution to the non-planar four-loop universal anomalous dimension, which can be rewritten in the terms of the usual harmonic sums as

$$\gamma_{uni,np,\zeta_3}^{(3)}(j) = -192 S_1 (S_1 S_{-2} + S_3) . \quad (26)$$

As we wrote in the beginning, in the course of calculations we can easily obtain the non-planar contribution to the first even moment of the four-loop non-singlet anomalous dimension of Wilson twist-2 operators in QCD. Formally, we should consider the following Wilson twist-2 operator (cf. Eq.(4))

$$\mathcal{O}_{\mu_1, \dots, \mu_j}^i = \hat{S} \bar{\psi} \Lambda^i \gamma_{\mu_1} \mathcal{D}_{\mu_2} \dots \mathcal{D}_{\mu_j} \psi , \quad i = 1, \dots, 8 , \quad (27)$$

where ψ is the quark field and Λ^i is the flavour group generator of $SU(N_F)$. This operator is multiplicative renormalizable, that is does not mix with all other Wilson twist-2 operators in QCD. To find its anomalous dimension we should calculate the matrix element of this operator sandwiched between quark states with the same flavour. For the first even moment ($j = 2$) of operator $\mathcal{O}_{\mu_1, \dots, \mu_j}^i$ from Eq.(27)

$$\mathcal{O}_{\mu_1, \mu_2} = \hat{S} \bar{\psi} \Lambda^i \gamma_{\mu_1} \mathcal{D}_{\mu_2} \psi \quad (28)$$

all necessary diagrams are included in our above calculations. But we should remember that quarks are in the fundamental representation of color group, while gauginos in $\mathcal{N} = 4$ SYM are in the adjoint representation. This leads to the appearance of additional quartic Casimir operators of the fundamental and adjoint representations (see Refs.[22] and [20])

$$\frac{d_F^{abcd} d_F^{abcd}}{N_A} = \frac{N_c^4 - 6N_c^2 + 18}{96N_c^2} , \quad (29)$$

$$\frac{d_F^{abcd} d_A^{abcd}}{N_A} = \frac{N_c(N_c^2 + 6)}{8} , \quad (30)$$

$$\frac{d_A^{abcd} d_A^{abcd}}{N_A} = d_{44} = \frac{N_c^2(N_c^2 + 36)}{24} \quad (31)$$

and for the color group $SU(N_c)$ the more simple Casimir operators are:

$$T_F = \frac{1}{2} , \quad C_F = \frac{N_c^2 - 1}{2N_c} , \quad C_A = N_c , \quad N_A = N_c^2 - 1 . \quad (32)$$

So, our result for the non-planar contribution to the four-loop non-singlet anomalous dimension of Wilson twist-2 operator (28) is given by

$$\gamma_{ns,np}^{(3)}(j=2) = 32 \left(13 + 16\zeta_3 - 40\zeta_5 \right) n_f \frac{d_F^{abcd} d_F^{abcd}}{N_F} - 8 \left(23 - 62\zeta_3 - 160\zeta_5 \right) \frac{d_F^{abcd} d_A^{abcd}}{N_F}, \quad (33)$$

where n_f is the number of active quarks and

$$\gamma_{ns} = \sum_{n=0}^{\infty} \gamma_{ns}^{(n)} a_s^{(n+1)}, \quad a_s = \frac{\alpha_s}{4\pi}. \quad (34)$$

We are going to complete this result with the rest parts in the future calculations. Unfortunately, the obtained result can not be compared with the existing result for the first even moment of non-singlet anomalous dimension from [11], as in the result from Ref.[11] all Casimir operators are written explicitly for QCD with three active quarks (i.e. for the gauge group $SU(3)$ with $n_f = 3$), so it is impossible to separate non-planar contribution from planar. Comparison with the result from Ref.[12] is impossible, because this result is proportional to $(n_f)^{i-1} a_s^i$, that is the number of possible quark loops, which is equal to three at four-loop order.

To conclude, we note that our guess about general form of ζ_3 contribution to the non-planar four-loop universal anomalous dimension of twist-2 operators in $\mathcal{N} = 4$ SYM theory (26) will be checked by calculations of the next even moments in our forthcoming calculations. We hope, that these new calculations together with the different constraints will give enough information for the reconstruction of the rational part to extract the non-planar scaling function. In any case our result (12) of the *full direct* calculations for the non-planar contribution to the four-loop anomalous dimension of twist-2 operator is new and can be used in the investigations of non-planar aspects of AdS/CFT-correspondence.

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